# On Multiuser Power Region of Fading Multiple-Access Channel with Multiple Antennas

Rui Zhang, Mehdi Mohseni, and John M. Cioffi



**Abstract**

This paper is concerned with the fading multiple-input multiple-output multiple-access channel (MIMO-MAC) with multiple receive antennas at the base station (BS) and multiple transmit antennas at each mobile terminal (MT). Two multiple-access techniques are considered for scheduling transmissions from each MT to the BS at the same frequency, which are *space-division multiple-access* (SDMA) and *time-division multiple-access* (TDMA). For SDMA, all MTs transmit simultaneously to the BS and their individual signals are resolved at the BS via multiple receive antennas while for TDMA, each MT transmits independently to the BS during mutually orthogonal time slots. It is assumed that the channel-state information (CSI) of the fading channel from each MT to the BS is *unknown* at each MT transmitter, but is perfectly *known* at the BS receiver. Thereby, the BS can acquire the long-term channel-distribution information (CDI) for each MT. This paper extends the well-known *transmit-covariance feedback* scheme for the point-to-point fading MIMO channel to the fading MIMO-MAC, whereby the BS jointly optimizes the transmit signal covariance matrices for all MTs based on their CDI, and then sends each transmit covariance matrix back to the corresponding MT via a feedback channel. The main goal of this paper is to characterize the so-called *multiuser power region* under the multiuser transmit-covariance feedback scheme for both SDMA and TDMA. The power region is defined as the constitution of all user transmit power-tuples that can achieve reliable transmissions for a given set of user target rates. Simulation results show that SDMA can achieve substantial power savings over TDMA for the fading MIMO-MAC, even when the number of antennas at the BS is equal to that at each MT. Moreover, this paper demonstrates the usefulness of the multiuser power region for maintaining *proportionally-fair* power consumption among the MTs.

**Index Terms**

Multiple-input multiple-output (MIMO), multi-antenna systems, Gaussian multiple-access channel (MAC), fading channel, capacity region, power region, partial channel feedback, space-division multiple-access (SDMA), time-division multiple-access (TDMA), proportional fairness, resource allocation, convex optimization.

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Rui Zhang is with the Institute for Infocomm Research, A*STAR, Singapore. (e-mail: rzhang@i2r.a-star.edu.sg)

Mehdi Moshseni is with the ASSIA, Inc. (e-mail:mmohseni@assia-inc.com)

John M. Cioffi is with the Department of Electrical Engineering, Stanford University. (e-mail: cioffi@stanford.edu)

# I. Introduction

Transmission through multiple transmit and multiple receive antennas, or the so-called multiple-input multiple-output (MIMO) technology, is known as an efficient means for providing enormous information rates in rich-scattering mobile environments [1]-[3]. Characterization of the fading MIMO channel capacity limits, under various assumptions on the transmitter-side and receiver-side channel-state information (CSI) and channel-distribution information (CDI), has motivated a great deal of valuable scholarly work (e.g., [4] and references therein). In particular, the case where the CSI is perfectly *known* at the receiver but *unknown* at the transmitter has drawn much interest due to its validity in many practical situations. This is because the presumption of perfect receiver-side CSI is usually reasonable for wireless channels where the receiver can locally estimate the fading channel, while the complete CSI feedback from the receiver to the transmitter is difficult or even impossible. Consequently, many schemes that exploit various forms of *partial* channel feedback have been proposed in literature. Among others, the *transmit-covariance feedback* scheme is known to be capable of achieving data rates close to the fading MIMO channel ergodic capacity when the channel CDI exhibits some long-term consistent statistical properties, e.g., constant channel mean and/or constant channel covariance matrix [5]-[8]. In the transmit-covariance feedback scheme, the receiver determines the transmit signal covariance matrix based on the CDI, and then sends it back to the transmitter through a feedback channel. In [5]-[8], the feedback transmit covariance matrix for optimizing the channel ergodic capacity, and the conditions under which the beamforming – the transmit covariance matrix has rank one – is optimal, have been established for the single-user fading channel. In this scheme, the transmit covariance matrix is fixed as long as the CDI is not changed. Therefore, this scheme requires much less feedback complexity and is also more robust to the delay of the feedback channel compared to other partial channel feedback schemes based on the instantaneous MIMO channel realizations (e.g., [9], [10] and references therein).

This paper considers the fading MAC with additive white Gaussian noise (AWGN) at the receiver, and assumes that the CSI from each mobile terminal (MT) to the base station (BS) is unknown at each MT transmitter, but is perfectly known at the BS receiver. Thus, the BS can acquire the channel CDI for each MT. This paper extends the transmit-covariance feedback scheme for the single-user fading MIMO

channel to the fading MIMO multiple-access channel (MIMO-MAC) where multiple antennas are used by the BS and possibly by each MT. Two multiple-access techniques are considered for scheduling transmissions from each MT to the BS at the same frequency: *space-division multiple-access* (SDMA) and *time-division multiple-access* (TDMA). For SDMA, all MTs transmit simultaneously to the BS and their individual signals are decoded jointly at the BS while for TDMA, each MT transmits independently to the BS during mutually orthogonal time slots and thus only single-user decoding is needed. The multiuser transmit-covariance feedback scheme is then described as follows. For SDMA, the BS first jointly optimizes the transmit signal covariance matrices for all MTs, based on the multiuser CDI as well as the rate requirement and the power budget of each MT, and then sends them back to each MT for transmission. This scheme has also been considered in [11], [12] for characterizing the capacity region and establishing the conditions for the optimality of beamforming for the fading MIMO-MAC, respectively. In contrast, for TDMA, the BS jointly optimizes the duration of transmission time slot for each MT along with their transmit covariance matrices. These optimized values are then sent back to each corresponding MT via the feedback channel.

This paper studies the information-theoretic limits of the fading MIMO-MAC under the multiuser transmit-covariance feedback scheme when either SDMA or TDMA is employed. Two commonly adopted means to measure the information-theoretic limits of multiuser channels are the *capacity region* and the *power region*. The capacity region is defined as the constitution of all achievable rate-tuples for the users given their individual power constraints, while the power region consists of all possible power-tuples for the users under which a given rate-tuple is achievable. This paper is mainly concerned with the characterization of the multiuser power region. Our motivations are justified as follows:

First, characterization of the power region for the fading MIMO-MAC is a challenging problem. Considering initially the case of SDMA, the capacity region of a *deterministic* (no fading) Gaussian MAC with a single transmit and a single receive antenna (SISO-MAC) has the well-known *polymatroid* structure [13], which also holds for the fading MIMO-MAC. On the other hand, the power region of a *deterministic* SISO-MAC is known to have a *contra-polymatroid* structure [14]. The polymatroid and the contra-polymatroid structures can be utilized to reduce significantly the computational complexity of

finding the boundary points of the capacity region and the power region, respectively [13], [14]. However, the contra-polymatroid structure is non-existent for the power region when the channel exhibits fading [15] and/or the BS uses multiple antennas [16].[1] As a result, characterization of the power region for the fading MIMO-MAC under SDMA is yet fully understood in literature. On the other hand, for TDMA, given the duration of each MT transmission time slot (e.g., equal time-slot durations for all MTs in the conventional TDMA), the BS only needs to optimize the transmit covariance matrices for the MTs independently such that their individual transmission powers are minimized for supporting their own target rates. However, with time-slot duration adjustment for each MT, the BS now needs to consider the more challenging problem of jointly optimizing the time-slot durations and the transmit covariance matrices for all MTs. This joint optimization has been less studied in literature.

Secondly, characterization of the power region can potentially provide very useful insights on the resource allocation problem for wireless networks, e.g., the wireless cellular network. In many practical situations, each BS in the cellular network controls the transmit power of each MT in its cell such that each individual MT rate demand – transmission quality-of-service (QoS) – is satisfied (e.g., [17], [18]). Power control can be beneficial in many aspects, e.g., to maintain fair power consumption among MTs, to tailor for each MT's peak-power constraint, and to mitigate the co-channel interference between multiple cells operating at the same frequency. By exploiting the multiuser power region, the minimum power consumption in the network can be achieved under practical transmission constraints.

The main contributions of this paper are summarized as follows:

- For both SDMA and TDMA, the paper presents efficient algorithms for characterizing each boundary point of the power region for the fading MIMO-MAC. The developed algorithms are based on a Lagrange primal-dual approach that exploits a novel dual relationship between the power region and the corresponding capacity region for the fading MIMO-MAC. For SDMA, the proposed algorithm determines jointly the optimal transmit covariance matrices for all MTs as well as their optimal decoding order at the receiver. For TDMA, all MT transmit covariance matrices along with their assigned variable time-slot durations are jointly optimized.

[1] More discussions on this aspect are postponed to Section IV-A.

- In addition to the conventional way to characterize the boundary of the power region by solving a sequence of user *weighted* sum-power minimization problems subject to fixed user rate constraints, this paper proposes an alternative means for such characterization by considering the user sum-power minimization problem under different *user power-profile* constraints, where the user power-profile regulates the power consumption of users under some given proportional fairness.

The remainder of this paper is organized as follows. Section II introduces the system model for the fading MIMO-MAC and describes the proposed multiuser transmit-covariance feedback scheme under SDMA and TDMA. Section III provides the definition of the power region for the fading MIMO-MAC, together with two problem formulations for characterization of the power region, based on the user weighted sum-power minimization and the user power-profile vector, respectively. Section IV and Section V study the power region in the case of SDMA and TDMA, respectively, and present efficient algorithms for characterizing the power region in each case. Section VI provides numerical results to verify the usefulness of the proposed algorithms under realistic channel parameters. Finally, Section VII concludes the paper.

*Notations*: Scalar signals are denoted by lower-case letters, e.g., $x, y$. Bold-face lower-case letters are used for vector signals, e.g., $\boldsymbol{x}, \boldsymbol{y}$, and bold-face upper-case letters for matrices, e.g., $\boldsymbol{S}$, $\boldsymbol{M}$. $|\boldsymbol{S}|$ denotes the determinant, $\boldsymbol{S}^{-1}$ the inverse and $\texttt{Tr}(\boldsymbol{S})$ the trace of a square matrix $\boldsymbol{S}$. For any general matrix $\boldsymbol{M}$, $\boldsymbol{M}^T$ and $\boldsymbol{M}^\dagger$ denote its transpose and conjugate transpose, respectively. $\boldsymbol{I}$ denotes the identity matrix. $\mathbb{E}[\cdot]$ denotes statistical expectation. $\mathbb{C}^{x\times y}$ denotes the space of $x \times y$ matrices with complex entries. $\mathbb{R}^M$ denotes the $M$-dimensional real Euclidean space and $\mathbb{R}^M_+$ is the nonnegative orthant. The distribution of a circular symmetric complex Gaussian (CSCG) vector with the mean vector $\boldsymbol{x}$, and the covariance matrix $\boldsymbol{\Sigma}$ is denoted by $\mathcal{CN}(\boldsymbol{x}, \boldsymbol{\Sigma})$, and $\sim$ means "distributed as." The sign $\succeq$ denotes the generalized inequality [19] and for a square matrix $\boldsymbol{S}$, $\boldsymbol{S} \succeq 0$ means that $\boldsymbol{S}$ is positive semi-definite. $\min(x, y)$ denotes the minimum between two real numbers $x$ and $y$.

## II. System Model

This paper considers a narrow-band flat fading MIMO-MAC with $r$ receive antennas at the BS and $K$ MTs equipped with $t_1, \ldots, t_K$ transmit antennas, respectively. All MTs transmit synchronously to

the BS by sharing a common frequency band. It is assumed that the space of fading states is continuous and infinite, and the fading process is stationary and ergodic. Under the assumption that the transmitted symbol period is equal to the inverse of the common transmission bandwidth for all MTs, at each fading state $\nu$, the equivalent discrete-time MAC is given by

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{H}_k(\nu)\boldsymbol{x}_k + \boldsymbol{z}, \tag{1}$$

where $\boldsymbol{y} \in \mathbb{C}^{r\times 1}$ denotes the received signal vector, $\boldsymbol{x}_k \in \mathbb{C}^{t_k\times 1}$ and $\boldsymbol{H}_k(\nu) \in \mathbb{C}^{r\times t_k}$ denote, respectively, the transmitted signal vector and the channel matrix of MT $k$, $k = 1, \ldots, K$; $\boldsymbol{z} \in \mathbb{C}^{r\times 1}$ denotes the vector of additive noise at the receiver, and it is assumed that $\boldsymbol{z} \sim \mathcal{CN}(0, \boldsymbol{I})$.

This channel model also assumes that the CSI is perfectly known at the BS but is unknown at each MT. With the CSI available, the BS can acquire the long-term CSI statistics (or equivalently, the CDI) of each MT. Based on the multiuser CDI, the BS determines the transmit signal covariance matrices for all MTs jointly according to their individual rate requirement and power budget, and then sends them back to each MT for transmission. This paper refers to this scheme as *multiuser transmit-covariance feedback*. Let the transmit covariance matrix of MT $k$ be $\boldsymbol{S}_k \triangleq \mathbb{E}[\boldsymbol{x}_k\boldsymbol{x}_k^\dagger]$, where the expectation is taken over the code-book and $\boldsymbol{S}_k \succeq 0$. $\boldsymbol{S}_k$ is assumed to be *fixed* for all fading states $\nu$. Fig. 1 illustrates the system model considered in this paper. Since this paper is concerned with the information-theoretic limits of a Gaussian MAC, the optimal Gaussian code-book is assumed for each MT, i.e., $\boldsymbol{x}_k \sim \mathcal{CN}(0, \boldsymbol{S}_k), \forall k$. The transmit covariance matrix of MT $k$ can be expressed by its eigenvalue decomposition as

$$\boldsymbol{S}_k = \boldsymbol{V}_k \boldsymbol{\Sigma}_k \boldsymbol{V}_k^\dagger. \tag{2}$$

$\boldsymbol{V}_k \in \mathbb{C}^{t_k\times d_k}$ is known as the *precoding matrix* where $\boldsymbol{V}_k^\dagger\boldsymbol{V}_k = \boldsymbol{I}$, and $d_k \leq \min(t_k, r)$. $d_k$ is usually referred to as the *spatial multiplexing gain* as it measures the number of degrees of transmission freedom (or equivalently, the number of data streams) in the spatial domain. If $d_k$ is equal to one, the associated transmission scheme is usually referred to as *beamforming*. $\boldsymbol{\Sigma}_k$ is a $d_k \times d_k$ diagonal matrix with positive diagonal elements that provide the *power-loading* to the corresponding transmitted data streams. The transmitter of each MT can be implemented as the cascade of the following operations: encoding the information bits by the optimal Gaussian code-book, interleaving coded symbols randomly into each data

stream, power-loading and then jointly precoding all data streams. Next, two multiple-access techniques considered in this paper are illustrated, namely, SDMA and TDMA.

For SDMA, all MTs transmit simultaneously to the BS. In this paper, it is assumed that the BS receiver uses the optimal (capacity-achieving) multiuser detection. For a fixed set of $\{\boldsymbol{S}_k\}$, $k = 1, \ldots, K$, all the rate-tuples in the set, $\mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k\})$, defined below, are achievable (e.g., [11], [12]):

$$\mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k\}) = \left\{ \boldsymbol{r} \in \mathbb{R}_+^K : \sum_{k \in \mathcal{J}} r_k \leq \mathbb{E}_\nu \left[ \frac{1}{2} \log \left| \sum_{k \in \mathcal{J}} \boldsymbol{H}_k(\nu) \boldsymbol{S}_k \boldsymbol{H}_k^\dagger(\nu) + \boldsymbol{I} \right| \right] \ \forall \mathcal{J} \subseteq \{1, \ldots, K\} \right\}. \quad (3)$$

The code-book of MT $k$ should satisfy $\mathtt{Tr}(\boldsymbol{S}_k) = p_k$, where $\boldsymbol{p} = (p_1, \ldots, p_K) \in \mathbb{R}_+^K$ denotes the vector of average transmit powers for the MTs.

On the other hand, for TDMA, the BS divides the total transmission time into multiple transmission blocks of equal duration $T$. Each transmission block is then further divided into $K$ non-overlapping time slots assigned to the MTs. These time slots are assumed fixed over all blocks. During the time slot of MT $k$, only this MT communicates with the BS and other MTs are silent, i.e., $\boldsymbol{x}_{k'} = 0, \forall k' \neq k$ in (1). Let $\tau_k T$ denote the time-slot duration assigned to MT $k$, where $0 \leq \tau_k \leq 1, \forall k$ and $\sum_{k=1}^K \tau_k = 1$. The BS determines jointly the slot duration for each MT and their transmit covariance matrices, and then sends them back to each MT. For a fixed set of $\{\tau_k\}$ and $\{\boldsymbol{S}_k\}$, $k = 1, \ldots, K$, each MT transmits over a single-user fading MIMO channel studied in e.g., [1], [2], and thus the following rate-tuples in the set, $\mathcal{C}_{\mathtt{TDMA}}(\{\tau_k\}, \{\boldsymbol{S}_k\})$, defined below are achievable:

$$\mathcal{C}_{\mathtt{TDMA}}(\{\tau_k\}, \{\boldsymbol{S}_k\}) = \left\{ \boldsymbol{r} \in \mathbb{R}_+^K : r_k \leq \tau_k \mathbb{E}_\nu \left[ \frac{1}{2} \log \left| \boldsymbol{H}_k(\nu) \frac{\boldsymbol{S}_k}{\tau_k} \boldsymbol{H}_k^\dagger(\nu) + \boldsymbol{I} \right| \right] \ \forall k \in \{1, \ldots, K\} \right\}. \quad (4)$$

And, again, $\boldsymbol{p} = (p_1, \ldots, p_K) \in \mathbb{R}_+^K$ denotes the average transmit powers for the MTs. Notice that for TDMA, the actual transmit power for MT $k$ during its assigned time-slot duration $\tau_k T$ is $\mathtt{Tr}\left(\boldsymbol{S}_k\right)/\tau_k$, but the average transmit power $p_k$ over each block duration $T$ is $\mathtt{Tr}(\boldsymbol{S}_k)$, the same as SDMA.

*Remark 2.1:* In this work, for both SDMA and TDMA, we consider each user's achievable rate in the "ergodic" sense, i.e., averaged over all ergodic fading states. In the case of fast-fading channel, the resultant ergodic capacity can be achievable by assigning each MT a constant-rate code-book for which the codeword length is sufficiently long so as to exploit the ergodicity of the channel. In contrast, in the case of slow-fading channel, each MT's codeword might not be able to span over all possible fading

states because of practical transmission delay constraint. However, if each MT uses multiple code-books with variable rates, the BS, based on the instantaneous channel, can determine the transmission rate of each MT and then sends back the corresponding code-book index to each MT for transmission. As in the fast-fading case, the same ergodic capacity (sometimes known as the expected capacity) can be achieved for each MT via time-averaging its transmission rates over different fading states.

## III. Power Region for Fading MIMO-MAC: Definitions and Characterizations

The multiuser *power region* for the fading MAC is defined as the constitution of all user power-tuples under which a given set of rates is achievable for all the MTs. Let $\boldsymbol{R} = (R_1, \ldots, R_K) \in \mathbb{R}_+^K$ denote the vector of rate requirements for the MTs. The power region is then defined as

$$\mathcal{P}_{\mathrm{SDMA}}(\boldsymbol{R}) \triangleq \left\{\boldsymbol{p} \in \mathbb{R}_+^K : \exists \ \{\boldsymbol{S}_k\}, \text{such that } \boldsymbol{R} \in \mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k\}), p_k = \mathtt{Tr}(\boldsymbol{S}_k), k = 1, \ldots, K\right\}, \quad (5)$$

for SDMA, and

$$\mathcal{P}_{\mathrm{TDMA}}(\boldsymbol{R}) \triangleq \left\{\boldsymbol{p} \in \mathbb{R}_+^K : \exists \ \{\tau_k\}, \{\boldsymbol{S}_k\}, \text{such that } \boldsymbol{R} \in \mathcal{C}_{\mathtt{TDMA}}(\{\tau_k\}, \{\boldsymbol{S}_k\}), p_k = \mathtt{Tr}(\boldsymbol{S}_k), k = 1, \ldots, K\right\}, \quad (6)$$

for TDMA. It is not hard to show that the power regions for both SDMA and TDMA are convex sets. The power region is illustrated in Fig. 2 for a two-user fading MAC under either SDMA or TDMA. The solid line in Fig. 2 represents the boundary points of the power region, which correspond to all *pareto* optimal power-tuples each minimizing a weighted-sum of the powers among the MTs. Each boundary point, e.g., point A as indicated in Fig. 2, might be characterized by two alternative means described as follows.

First, because of the convexity of the power region, each boundary point can be expressed as the solution to a *weighted sum-power minimization* (W-SPmin) problem stated below, for some nonnegative user weights, $\lambda_k$, $k = 1, \ldots, K$. For SDMA, the W-SPmin problem can be expressed as

*Problem 1:*

$$\begin{aligned} \underset{\{r_k\}, \{\boldsymbol{S}_k\}}{\mathtt{Minimize}} \quad & \sum_{k=1}^{K} \lambda_k \mathtt{Tr}(\boldsymbol{S}_k) && (7) \\ \mathtt{Subject\ to} \quad & r_k \geq R_k \quad \forall k && (8) \\ & \boldsymbol{r} \in \mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k\}) && (9) \\ & \boldsymbol{S}_k \succeq 0 \quad \forall k. && (10) \end{aligned}$$

Notice that $\{r_k\}$ are auxiliary variables. For TDMA, this W-SPmin problem is given by

<u>*Problem*</u> *2:*

$$\begin{aligned}
\underset{\{r_k\},\{\boldsymbol{S}_k\},\{\tau_k\}}{\texttt{Minimize}} \quad & \sum_{k=1}^{K} \lambda_k \texttt{Tr}\left(\boldsymbol{S}_k\right) && (11)\\
\texttt{Subject to} \quad & r_k \geq R_k \quad \forall k && (12)\\
& \boldsymbol{r} \in \mathcal{C}_{\texttt{TDMA}}(\{\tau_k\}, \{\boldsymbol{S}_k\}) && (13)\\
& \tau_k \geq 0 \quad \forall k && (14)\\
& \sum_{k=1}^{K} \tau_k = 1 && (15)\\
& \boldsymbol{S}_k \succeq 0 \quad \forall k. && (16)
\end{aligned}$$

By definition of the power region for a given target rate-tuple $\boldsymbol{R}$, each boundary point of the power region, $\boldsymbol{p}^*$, under either SDMA or TDMA, can be expressed as a power-tuple supporting the target rates in $\boldsymbol{R}$ that has the minimum weighted-sum, $\sum_{k=1}^{K} \lambda_k p_k^*$, for some weight vector $\boldsymbol{\lambda}$, among all the power-tuples that can support $\boldsymbol{R}$. Alternatively, the connection between the power-tuple $\boldsymbol{p}^*$ on the boundary of the power region and the target rate $\boldsymbol{R}$ can be established by employing the capacity region concept. Fig. 3 gives an illustration for this important observation. Consider the power region of a two-user MAC under the rate constraint $(R_1, R_2)$, as shown in Fig. 3 (a). Given $\lambda_1$ and $\lambda_2$, the solution to the W-SPmin problem is denoted by $(p_1^*, p_2^*)$, which is represented by point A in Fig. 3 (a), and satisfies $\lambda_1 p_1^* + \lambda_2 p_2^* = p^*$, where $p^*$ is the minimum value of the W-SPmin problem. Thus, the required minimum power-pair is $(p_1^*, p_2^*)$ for $(R_1, R_2)$. On the other hand, it is not hard to verify that the rate-pair $(R_1, R_2)$ is on the boundary of the capacity region for the same MAC under the *weighted sum-power constraint* given by $\lambda_1 p_1 + \lambda_2 p_2 \leq p^*$. This is shown by point B in Fig. 3 (b). Moreover, because of the convexity of the capacity region, $(R_1, R_2)$ must be the solution to a *weighted sum-rate maximization* (W-SRmax) problem for a given nonnegative user weight vector $\boldsymbol{\rho}$, as shown in Fig. 3 (b). The above observation has an important consequence, i.e., each power-region boundary point as the solution to the W-SPmin problem can be equivalently characterized as a boundary point of the corresponding capacity region under the same weighted sum-power constraint. As will be shown later in this paper, this result also motivates the proposed algorithms for the W-SPmin problem under both SDMA and TDMA.

Alternatively, each boundary point of the power region can also be considered geometrically as the

intersection of a line passing through the origin (the power-tuple with all zeros) and the boundary of the power region (see point A in Fig. 2). Let each line be characterized as $p_k = \alpha_k P$, for $k = 1, \ldots, K$ and $P \geq 0$. The vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K) \in \mathbb{R}_+^K$ is referred to as the *user power-profile* vector in this paper, and it is assumed that $\sum_{k=1}^K \alpha_k = 1$. The point where the line specified by $\boldsymbol{\alpha}$ intersects the power region boundary can be then obtained as the solution to the following optimization problem, referred to as the *sum-power minimization under the power-profile constraint* (SPmin-PPC). For SDMA, this problem can be expressed as

*Problem 3:*

$$\begin{aligned} \underset{P,\{r_k\},\{\boldsymbol{S}_k\}}{\texttt{Minimize}} \quad & P && (17)\\ \texttt{Subject to} \quad & r_k \geq R_k \;\; \forall k && (18)\\ & \boldsymbol{r} \in \mathcal{C}_{\texttt{SDMA}}(\{\boldsymbol{S}_k\}) && (19)\\ & \boldsymbol{S}_k \succeq 0 \;\; \forall k && (20)\\ & \texttt{Tr}\,(\boldsymbol{S}_k) \leq \alpha_k P \;\; \forall k && (21)\\ & P \geq 0. && (22) \end{aligned}$$

And similarly, the SPmin-PPC problem can be defined for TDMA. The characterization of the power region via some prescribed power-profile vector is useful for the BS to regulate the power consumption of MTs in a desired *proportionally-fair* manner.

It is not hard to show that both problems, W-SPmin and SPmin-PPC, are convex for either SDMA or TDMA, and hence, they can be solved by applying convex optimization techniques. The following two sections present the solutions to these problems for SDMA and TDMA, respectively.

## IV. Characterization of Power Region for SDMA

### A. Solutions to Weighted Sum-Power Minimization

This subsection considers Problem 1, the W-SPmin problem in the case of SDMA. For the special case of a deterministic (no fading) SISO-MAC, the W-SPmin can be simplified using the contra-polymatroid structure as proposed in [13], [14]. However, as shown later in this subsection, the approach in [13], [14] can not be applied here to handle the more general case of the fading MIMO-MAC. Thus, an

alterative approach is proposed.

**Deterministic SISO-MAC:** Consider a deterministic SISO-MAC where the channel gain for each MT in (1) is a positive constant, i.e., $|\boldsymbol{H}_k(\nu)| = \sqrt{h_k}, \forall \nu$. In this case, the solutions to the W-SPmin problem are obtained as follows.

*Theorem 1:* For a deterministic SISO-MAC consisting of $K$ users with channel gains $h_1, \ldots, h_K$, and rate requirements $R_1, \ldots, R_K$, the solutions to the W-SPmin problem under SDMA are given by [14, Lemma 3.2]:

$$p^*_{\pi(k)} = \begin{cases} \frac{\exp\left(2R_{\pi(1)}\right)-1}{h_{\pi(1)}} & \texttt{if } k=1 \\ \frac{\exp\left(2\sum_{i=1}^{k} R_{\pi(i)}\right)-\exp\left(2\sum_{i=1}^{k-1} R_{\pi(i)}\right)}{h_{\pi(k)}} & k=2,\ldots,K, \end{cases} \tag{23}$$

where the permutation $\pi$ indicates the optimal decoding order (user $\pi(1)$ is decoded last and user $\pi(K)$ is decoded first) at the receiver according to

$$\frac{\lambda_{\pi(1)}}{h_{\pi(1)}} \geq \cdots \geq \frac{\lambda_{\pi(K)}}{h_{\pi(K)}}. \tag{24}$$

Fig. 4 shows the connection between each boundary point on the power region and the corresponding capacity region earlier described in Section III for a two-user deterministic SISO-MAC. Fig. 4 (a) shows the power region given rate constraint $(R_1, R_2)$. Consider a vertex $(p_1^*, p_2^*)$ of this power region, and arbitrary positive weights $\lambda_1, \lambda_2$ such that $\frac{\lambda_1}{h_1} > \frac{\lambda_2}{h_2}$. According to Theorem 1, $(p_1^*, p_2^*)$ is the optimal solution to the W-SPmin problem, and is achievable by decoding order $2 \rightarrow 1$ (user 2's message is decoded before user 1's). For this given choice of $\lambda_1$ and $\lambda_2$, the boundary curve of the capacity region under the *weighted-sum power constraint* $\lambda_1 p_1 + \lambda_2 p_2 \leq p^*$, shown in Fig. 4 (b), can be represented as the union of all rate regions, $\mathcal{C}_{\texttt{SDMA}}(\{p_k\})$ defined in (3), over all values of $p_1$ and $p_2$ that satisfy $\lambda_1 p_1 + \lambda_2 p_2 = p^*$. Each $\mathcal{C}_{\texttt{SDMA}}(\{p_k\})$ is a pentagon with two vertices, each corresponding to one of the two possible decoding orders among the users [20]. The fact that user 2's message must be decoded first to achieve $(R_1, R_2)$ can be justified by the following observation. It is seen that the decoding order $2 \rightarrow 1$ always achieves higher rates for both users than the other decoding order $1 \rightarrow 2$, hence, it must be the optimal decoding order to achieve the rate target $(R_1, R_2)$ on the boundary of this capacity region. In general, for a deterministic SISO-MAC, all rate-tuples on the boundary of the capacity region under a weighted sum-power constraint can be achieved by a *unique* decoding order for each user. This

result is consistent with Theorem 1, i.e., given $\lambda_k$'s and $h_k$'s, the optimal decoding order of users can be first resolved by (24), and then their minimum powers can be found by (23).

**Fading MIMO-MAC:** Unfortunately, the contra-polymatroid structure is non-existent for the MAC when there is fading (e.g., the fading SISO-MAC) or there are multiple antennas at the receiver (e.g., the deterministic SIMO-MAC), and hence, it is non-existent for the general fading MIMO-MAC. Equivalently, the rate-tuples on the boundary of the capacity region under the weighted sum-power constraint for a fading MIMO-MAC, unlike the case of a deterministic SISO-MAC, might not correspond to a unique decoding order for each user. Fig. 5 illustrates this fact by showing the capacity region of a two-user fading SIMO-MAC ($t_1 = t_2 = 1, r = 2$) under a sum-power constraint $p_1 + p_2 \leq 10$ (i.e., $\lambda_1 = \lambda_2 = 1$). In this case, the channels $\{\boldsymbol{H}_1(\nu)\}$ and $\{\boldsymbol{H}_2(\nu)\}$ are assumed to be independent vectors each having independent entries distributed as $\mathcal{CN}(0, 1)$. The capacity region for this case is shown to be symmetric. The dashed line and the dotted line show how two vertices of the constituting $\mathcal{C}_{\mathtt{SDMA}}(\{p_k\})$ sweep on the boundary of the capacity region as $p_1$ and $p_2$ vary while their sum is kept equal to 10. It is observed that the boundary rate-tuples of this capacity region indeed correspond to different decoding orders; e.g., point A corresponds to the decoding order $1 \rightarrow 2$, while point D does for $2 \rightarrow 1$. There is also a part of the capacity region (e.g., point B is in this region) that does not consist of any vertices. This part of the region is referred to as the *time-sharing* region and consists of the -45 degree boundary lines of the constituting $\mathcal{C}_{\mathtt{SDMA}}(\{p_k\})$. Hence, any point in the time-sharing region is not achievable by successive decoding given any fixed decoding orders, and time-sharing the transmission rates and the decoding orders among two users is required. As a result, unlike the deterministic SISO-MAC, given $\lambda_k$'s and $\boldsymbol{H}_k(\nu)$'s, the optimal decoding order for the W-SPmin problem can not be resolved directly for this channel.

One heuristic approach (e.g., [21]-[23]) for solving W-SPmin problem for the fading MIMO-MAC under SDMA might be searching through all possible $K!$ decoding orders and then finding the optimal decoding order that gives the minimum weighted sum-power to support the target rates. This approach might work for some special cases, but as explained below, is problematic in general.

For any fixed decoding order $\pi$, Problem 1 can be written as

$$\underset{\{\boldsymbol{S}_k\}}{\texttt{Minimize}} \quad \sum_{k=1}^{K} \lambda_k \texttt{Tr}\left(\boldsymbol{S}_k\right) \tag{25}$$

$$\texttt{Subject to} \quad \mathbb{E}_{\nu}\left[\frac{1}{2}\log\frac{\left|\sum_{i=1}^{k}\boldsymbol{H}_{\pi(i)}(\nu)\boldsymbol{S}_{\pi(i)}\boldsymbol{H}_{\pi(i)}^{\dagger}(\nu)+\boldsymbol{I}\right|}{\left|\sum_{i=1}^{k-1}\boldsymbol{H}_{\pi(i)}(\nu)\boldsymbol{S}_{\pi(i)}\boldsymbol{H}_{\pi(i)}^{\dagger}(\nu)+\boldsymbol{I}\right|}\right] \geq R_{\pi(k)} \quad \forall k \tag{26}$$

$$\boldsymbol{S}_k \succeq 0 \quad \forall k, \tag{27}$$

where the left-hand-side (LHS) of (26) is the achievable rate for MT $\pi(k)$, denoted as $r_{\pi(k)}^{(\pi)}$, under the decoding order $\pi$. Except for MT $\pi(1)$, the rate constraints in (26) are not convex, rendering the above optimization problem non-convex in general and, hence, it can not be solved efficiently. A suboptimal method that approximately solves this problem is described as follows. Starting from the last decoded MT $\pi(1)$, the method minimizes the power required to maintain the target rate for that MT, while considering MTs that have yet been decoded as interference. For example, for the two-user case and the decoding order of MT 2 followed by MT 1, the method first determines $\boldsymbol{S}_1$ with the minimum $p_1$ that satisfies $\mathbb{E}_{\nu}[\frac{1}{2}\log|\boldsymbol{H}_1(\nu)\boldsymbol{S}_1\boldsymbol{H}_1^{\dagger}(\nu)+\boldsymbol{I}|] \geq R_1$ and then fixes $\boldsymbol{S}_1$ and determines $\boldsymbol{S}_2$ with the minimum power $p_2$ that satisfies $\mathbb{E}_{\nu}[\frac{1}{2}\log|\boldsymbol{H}_1(\nu)\boldsymbol{S}_1\boldsymbol{H}_1^{\dagger}(\nu)+\boldsymbol{H}_2(\nu)\boldsymbol{S}_2\boldsymbol{H}_2^{\dagger}(\nu)+\boldsymbol{I}|] \geq R_1+R_2$. Each of these two optimizations are convex and, hence, they both can be solved efficiently. The above algorithm is referred to as the *greedy* algorithm since each MT simply minimizes its own transmit power. For the special case of the fading SISO-MAC and SIMO-MAC where each $\boldsymbol{S}_k$ is simply a scalar and is equal to $p_k$, the obtained $p_1$ and $p_2$ via the greedy algorithm are in fact optimal and minimize the weighted sum-power for any weights $\boldsymbol{\lambda}$ under the given decoding order. This is because from (26) it can be shown that the minimum $p_{\pi(k)}$ required to support $R_{\pi(k)}$ for user $\pi(k)$ is an increasing function of the powers for the not-yet-decoded users, $p_{\pi(1)}, \ldots, p_{\pi(k-1)}$. However, for the general fading MIMO-MAC, the above greedy algorithm might not be optimal because each MT now can adjust its covariance matrix to balance between minimizing its own transmit power and reducing the interference it causes to the users decoded earlier in the order.

Nevertheless, even if the W-SPmin problem can be solved for each decoding order, the obtained powers that have the minimum weighted-sum among all decoding orders might still be suboptimal. This can occur when the target rate-tuple does not correspond to a unique optimal decoding order, e.g.,

the rate-pair B in Fig. 5 that is on the boundary of the time-sharing region.

From the above discussions, it follows that for the fading MIMO-MAC in general, the decoding order and the transmit covariance matrices for the MTs need to be jointly optimized for solving the W-SPmin problem under SDMA. This motivates the algorithm presented next.

**Proposed Algorithm:** The proposed algorithm for Problem 1 is based on its Lagrangian [19], which is given below:

$$\mathcal{L}(\{\boldsymbol{S}_k\}, \{r_k\}, \boldsymbol{\mu}) = \sum_{k=1}^{K} \lambda_k \mathtt{Tr}(\boldsymbol{S}_k) - \sum_{k=1}^{K} \mu_k (r_k - R_k), \tag{28}$$

where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_K) \in \mathbb{R}_+^K$ denotes the vector of dual variables associated with the rate inequality constraints in (8). The variables, $\{\boldsymbol{S}_k\}$ and $\{r_k\}$ belong to the set denoted by $\mathcal{D}$, which is specified by the remaining constrains in (9) and (10). Then the Lagrange dual function [19] is defined as

$$g(\boldsymbol{\mu}) = \min_{\{\boldsymbol{S}_k, r_k\} \in \mathcal{D}} \mathcal{L}(\{\boldsymbol{S}_k\}, \{r_k\}, \boldsymbol{\mu}). \tag{29}$$

The dual function serves as a lower bound on the optimal value of the original (primal) problem, denoted by $q^*$, i.e., $q^* \geq g(\boldsymbol{\mu}), \forall \boldsymbol{\mu}$ [19]. The dual problem is then defined as $\max_{\boldsymbol{\mu} \succeq 0} g(\boldsymbol{\mu})$ [19]. Let the optimal value of the dual problem be denoted by $d^*$ that is achievable by the optimal dual variables $\boldsymbol{\mu}^*$, i.e., $d^* = g(\boldsymbol{\mu}^*)$. For a convex optimization problem, the Slater's condition states that the duality gap, $q^* - d^* \geq 0$, is indeed zero if the primal problem has a feasible solution set [19]. By using sufficiently large user powers, the set $\mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k\})$ in (9) can be made arbitrarily large to contain any finite rate target $\boldsymbol{R}$ as an interior point. In other words, we can always find a feasible set $\{\boldsymbol{S}_k\}$ that satisfies any given rate constraint $\boldsymbol{R}$ for Problem 1. Thus, the Slater's condition holds and the duality gap is zero for Problem 1. This result suggests that $q^*$ can be obtained by first minimizing the Lagrangian $\mathcal{L}$ to obtain the dual function $g(\boldsymbol{\mu})$ for some given $\boldsymbol{\mu}$, and then maximizing $g(\boldsymbol{\mu})$ over all possible values $\boldsymbol{\mu}$.

First, consider the minimization of $\mathcal{L}$ to obtain the dual function $g(\boldsymbol{\mu})$. In this case, $\boldsymbol{\mu}$ is fixed and the variables are $\{\boldsymbol{S}_k\}$ and $\{r_k\}$. From (28), it follows that the minimization of $\mathcal{L}$ can be rewritten as

the following equivalent problem:

$$\underset{\{r_k\},\{\boldsymbol{S}_k\}}{\texttt{Minimize}} \quad \sum_{k=1}^{K}\lambda_k \texttt{Tr}(\boldsymbol{S}_k) - \sum_{k=1}^{K}\mu_k r_k \tag{30}$$

$$\texttt{Subject to} \quad \boldsymbol{r} \in \mathcal{C}_{\texttt{SDMA}}(\{\boldsymbol{S}_k\}) \tag{31}$$

$$\boldsymbol{S}_k \succeq 0 \quad \forall k. \tag{32}$$

From the definition of $\mathcal{C}_{\texttt{SDMA}}$ in (3), there are $2^K - 1$ rate constraints implied by (31), which are difficult to be incorporated directly into the optimization. In order to simplify the problem, the following theorem is utilized to remove these constraints in (31):

*Theorem 2:* For any permutation $\pi$ over $\{1,\ldots,K\}$ and a fixed set of covariance matrices $\{\boldsymbol{S}_k\}$, $\boldsymbol{r}^{(\pi)}$ defined as

$$r^{(\pi)}_{\pi(k)} = \mathbb{E}_\nu\left[\frac{1}{2}\log\frac{\left|\sum_{i=1}^{k}\boldsymbol{H}_{\pi(i)}(\nu)\boldsymbol{S}_{\pi(i)}\boldsymbol{H}^\dagger_{\pi(i)}(\nu)+\boldsymbol{I}\right|}{\left|\sum_{i=1}^{k-1}\boldsymbol{H}_{\pi(i)}(\nu)\boldsymbol{S}_{\pi(i)}\boldsymbol{H}^\dagger_{\pi(i)}(\nu)+\boldsymbol{I}\right|}\right] \tag{33}$$

is a *vertex* of the polymatroid $\mathcal{C}_{\texttt{SDMA}}(\{\boldsymbol{S}_k\})$ in $\mathbb{R}^K_+$. Furthermore, for any $\boldsymbol{\rho} \succeq 0$, the solution to the following W-SRmax problem:

$$\underset{\{r_k\}}{\texttt{Maximize}} \quad \sum_{k=1}^{K}\rho_k r_k \tag{34}$$

$$\texttt{Subject to} \quad \boldsymbol{r} \in \mathcal{C}_{\texttt{SDMA}}(\{\boldsymbol{S}_k\}), \tag{35}$$

is attained by a vertex $\boldsymbol{r}^{(\pi^*)}$, where $\pi^*$ is such that $\rho_{\pi^*(1)} \geq \rho_{\pi^*(2)} \geq \ldots \geq \rho_{\pi^*(K)}$.

*Proof:* Please refer to [13, Lemma 3.10]. ∎

Notice that Theorem 2 holds for any given $\{\boldsymbol{S}_k\}$. Furthermore, since minimization of $-\sum_k \mu_k r_k$ is equivalent to maximization of $\sum_k \mu_k r_k$, using Theorem 2, the rate constraints in (31) can be safely removed and the problem in (30) can be simplified as

$$\underset{\{\boldsymbol{S}_k\}}{\texttt{Minimize}} \quad \sum_{k=1}^{K}\lambda_k\texttt{Tr}(\boldsymbol{S}_k) - \sum_{k=1}^{K}\mu_{\pi(k)}\mathbb{E}_\nu\left[\frac{1}{2}\log\frac{\left|\sum_{i=1}^{k}\boldsymbol{H}_{\pi(i)}(\nu)\boldsymbol{S}_{\pi(i)}\boldsymbol{H}^\dagger_{\pi(i)}(\nu)+\boldsymbol{I}\right|}{\left|\sum_{i=1}^{k-1}\boldsymbol{H}_{\pi(i)}(\nu)\boldsymbol{S}_{\pi(i)}\boldsymbol{H}^\dagger_{\pi(i)}(\nu)+\boldsymbol{I}\right|}\right] \tag{36}$$

$$\texttt{Subject to} \quad \boldsymbol{S}_k \succeq 0 \quad \forall k, \tag{37}$$

where $\pi$ is a permutation such that $\mu_{\pi(1)} \geq \cdots \geq \mu_{\pi(K)}$. By rearranging the terms regarding user rates in (36), the above problem becomes the minimization of

$$\sum_{k=1}^{K}\lambda_k\texttt{Tr}(\boldsymbol{S}_k) - \sum_{k=1}^{K}\left(\mu_{\pi(k)} - \mu_{\pi(k+1)}\right)\mathbb{E}_\nu\left[\frac{1}{2}\log\left|\sum_{i=1}^{k}\left(\boldsymbol{H}_{\pi(i)}(\nu)\boldsymbol{S}_{\pi(i)}\boldsymbol{H}^\dagger_{\pi(i)}(\nu)\right)+\boldsymbol{I}\right|\right], \tag{38}$$

with only optimization variables $\boldsymbol{S}_k \succeq 0$, $\forall k$, and $\mu_{\pi(K+1)} \triangleq 0$. Since the above problem is convex with a twice differentiable objective function and positive semi-definite constraints, it can be solved numerically, e.g., by the interior-point method [19].

Next, the dual function $g(\boldsymbol{\mu})$ is maximized over all possible values $\boldsymbol{\mu}$. The search of $\boldsymbol{\mu}$ towards its optimal value $\boldsymbol{\mu}^*$ can be done, e.g., by the ellipsoid method [24], which utilizes the fact that the vector $\boldsymbol{\theta}$, defined as $\theta_k = R_k - r'_k$ for $k = 1, \ldots, K$, is a *sub-gradient* of $g(\boldsymbol{\mu})$ at any $\boldsymbol{\mu}$, where $\{\boldsymbol{S}'_k\}$ and $\{r'_k\}$ are the minimizers of $\mathcal{L}(\{\boldsymbol{S}_k\}, \{r_k\}, \boldsymbol{\mu})$ obtained via solving (38), i.e., $\mathcal{L}(\{\boldsymbol{S}'_k\}, \{r'_k\}, \boldsymbol{\mu}) = g(\boldsymbol{\mu})$.

*Remark 4.1:* It is noted that the algorithm proposed in [13, Algorithm 5.3] can also be modified to determine $\boldsymbol{\mu}^*$ for the problem at hand. However, from programming implementations, it is observed that this method may exhibit oscillation when some of $\mu_k^*$'s happen to be equal. In contrast, the ellipsoid method is more suitable because of its robust and superior convergence behavior.

The complete algorithm for Problem 1 in the case of SDMA is summarized below.

*Algorithm 1:*

- **Given** an ellipsoid $\mathcal{E}[0] \subseteq \mathbb{R}^K$, centered at $\boldsymbol{\mu}[0]$ and containing the optimal dual solution $\boldsymbol{\mu}^*$.
- **Set** $i = 0$.
- **Repeat**
  1. For given $\boldsymbol{\mu}[i]$, solve the problem given in (38) to obtain an optimal solution set $\{\boldsymbol{S}_k[i]\}$ and $\{r_k[i]\}$ that minimizes $\mathcal{L}(\{\boldsymbol{S}_k\}, \{r_k\}, \boldsymbol{\mu}[i])$ over $\mathcal{D}$;
  2. Update the ellipsoid $\mathcal{E}[i+1]$ based on $\mathcal{E}[i]$ and the sub-gradients $\theta_k[i] = R_k - r_k[i], k = 1, \ldots, K$. Set $\boldsymbol{\mu}[i+1]$ as the center of the new ellipsoid $\mathcal{E}[i+1]$;[2]
  3. Set $i \leftarrow i + 1$.
- **Until** the stopping criteria for the ellipsoid method is met.

One possible method to obtain the initial ellipsoid $\mathcal{E}[0]$ that contains the optimal dual solution $\boldsymbol{\mu}^*$ is given in Appendix I.

The primal-dual approach used for solving Problem 1 can be explained by the connection between the power region and the capacity region as described in Section III. In Fig. 3, it is observed that

[2]Notice that when locating the center of a new ellipsoid, we need to add the constraint that $\boldsymbol{\mu} \succeq 0$.

each power-tuple on the boundary of the power region for a given target rate-tuple, as the solution to Problem 1 for a given weight vector $\boldsymbol{\lambda}$, defines a capacity region that contains the target rate-tuple on its boundary. Consequently, the target rate-tuple can be expressed as the solution to a W-SRmax problem for some unknown user weight vector $\boldsymbol{\rho}$ over this capacity region. Clearly, the primal-dual approach establishes the above connection by finding the optimal dual variable $\boldsymbol{\mu}^*$ that is simply one candidate for the unknown weight vector $\boldsymbol{\rho}$ in this W-SRmax problem. From (38), it follows that the optimal decoding order of each MT is given by the magnitude of $\mu_k^*$, i.e., the optimal decoding order $\pi$ satisfies $\mu^*_{\pi(1)} \geq \cdots \geq \mu^*_{\pi(K)}$. Hence, the proposed algorithm successfully jointly optimizes the decoding order and the transmit covariance matrices of the MTs by exploiting the duality between the power region and the capacity region.

**Uniqueness of Solutions:** So far, the proposed algorithm determines the optimal value of the primal problem $q^*$ (equal to that of the dual problem $d^*$), the corresponding primal variables $\{\boldsymbol{S}_k^*\}$ and $\{r_k^*\}$ (it is yet claimed that these primal variables are the primal optimal solutions), and the dual optimal solutions $\boldsymbol{\mu}^*$ that satisfy,

$$q^* = d^* = \sum_{k=1}^{K} \lambda_k \mathtt{Tr}(\boldsymbol{S}_k^*) - \sum_{k=1}^{K} \mu_k^*(r_k^* - R_k). \tag{39}$$

In the following, the issue on the uniqueness of these solutions is addressed. While uniqueness of the dual optimal variables $\boldsymbol{\mu}^*$ is not an issue in the convergence of the proposed algorithm,[3] uniqueness of the primal variables, $\{\boldsymbol{S}_k^*\}$ and $\{r_k^*\}$, plays an important role in obtaining valid solutions for the W-SPmin problem. Since a primal-dual approach is used, the obtained primal variables that minimize the Lagrangian at $\boldsymbol{\mu}^*$ might not satisfy the rate constraints in (8). Notice that these variables are minimizers of the Lagrangian and are not necessarily the primal optimal solutions. However, according to Karush-Kuhn-Tucker (KKT) optimality conditions [19], the primal optimal solutions also minimize the Lagrangian at $\boldsymbol{\mu}^*$. Hence, if these Lagrangian minimizers can be proven to be unique, it follows that they satisfy the rate constraints in (8) automatically.

[3] By the primal-dual approach and the connection between the power region and the capacity region, it follows that $\boldsymbol{\mu}^*$ can be viewed as the weight vector $\boldsymbol{\rho}$ that attains the given rate requirements $\boldsymbol{R}$ as the solution to the W-SRmax problem over the corresponding capacity region. For the fading MIMO-MAC, as shown in Fig. 5, in general there is no "sharp" vertex with multiple tangent lines on the boundary of the capacity region under a weighted sum-power constraint. As a result, it can be inferred that $\boldsymbol{\mu}^*$ is unique for any rate-tuple on the boundary and, hence, the uniqueness of $\boldsymbol{\mu}^*$ is in general ensured.

*Theorem 3:* The primal optimal solutions $\{\boldsymbol{S}_k^*\}$ for Problem 1 under SDMA is unique.

*Proof:* Please refer to Appendix II. ■

If all $\mu_k^*$'s are positive and distinct, $\boldsymbol{r}^*$ (e.g., shown by Point A in Fig. 5) that maximizes $\sum_k \mu_k^* r_k$ over $\mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k^*\})$ will be one unique vertex of $\mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k^*\})$, which itself is also unique according to Theorem 3. In this case, from the KKT conditions, $r_k^*$'s automatically satisfy the rate constraints in (8). However, if $\mu_k^*$'s in some subset $\mathcal{J} \subseteq \{1, \ldots, K\}$ are positive and equal, $r_k^*$'s for the users in the set $\mathcal{J}$ may not be unique and consequently they may not satisfy the rate constraints in (8). This can be shown by point B, C and D in Fig. 5 where $\mu_1^* = \mu_2^*$. Any point on the straight line between point C and D maximizes $\sum_k \mu_k^* r_k$ over $\mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k^*\})$ because all these rate-pairs have the same sum-rate. Hence, if point B is the rate demand for our problem, because the simplified optimization in (38) always tends to use a vertex of the capacity region as the solution for $\boldsymbol{r}^*$, the proposed algorithm would converge to either point C or D as $\boldsymbol{r}^*$, which clearly does not satisfy the rate constraints. However, this is not an issue in the convergence of the proposed algorithm. As far as $\{\boldsymbol{S}_k^*\}$ is unique, the target rate-pair is ensued to be some convex combination of at most $K$ vertices of the unique $\mathcal{C}_{\mathtt{SDMA}}(\{\boldsymbol{S}_k^*\})$.

### *B. Solutions to Sum-Power Minimization Under Power-Profile Constraint*

This subsection presents the solutions to Problem 3, the SPmin-PPC problem under SDMA, where the transmit power of each MT is regulated according to a given power-profile vector $\boldsymbol{\alpha}$. The proposed algorithm is also based on a Lagrange primal-dual approach. The Lagrangian of the primal problem in (17) can be written as

$$\mathcal{L}(P, \{\boldsymbol{S}_k\}, \{r_k\}, \boldsymbol{\mu}, \boldsymbol{\delta}) = P + \sum_{k=1}^{K} \delta_k(\mathtt{Tr}(\boldsymbol{S}_k) - \alpha_k P) - \sum_{k=1}^{K} \mu_k(r_k - R_k), \tag{40}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K) \in \mathbb{R}_+^K$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_K) \in \mathbb{R}_+^K$ are dual variables associated with the inequality constraints in (18) and (21), respectively. Let $\mathcal{F}$ denote the set of primal variables specified by the remaining constraints in (19), (20) and (22), the Lagrange dual function can be then expressed as

$$g(\boldsymbol{\mu}, \boldsymbol{\delta}) = \min_{\{P, \boldsymbol{S}_k, r_k\} \in \mathcal{F}} P\left(1 - \sum_{k=1}^{K} \delta_k \alpha_k\right) + \sum_{k=1}^{K} \delta_k \mathtt{Tr}(\boldsymbol{S}_k) - \sum_{k=1}^{K} \mu_k(r_k - R_k). \tag{41}$$

From (41), it is necessary that $1 - \sum_{k=1}^{K} \delta_k \alpha_k \geq 0$ for the dual function to be bounded from below. In the case of $1 - \sum_{k=1}^{K} \delta_k \alpha_k = 0$, the optimal $P$ that minimizes the Lagrangian over $\mathcal{F}$ can take any positive value; while in the case of $1 - \sum_{k=1}^{K} \delta_k \alpha_k > 0$, the optimal $P$ must be equal to zero. Thus, in both case, the term associated with $P$ in (41) is indeed zero and, hence, can be removed from this point onward. The optimal value of $P$ can be then obtained as

$$P^* = \max_{\boldsymbol{\mu} \succeq 0, \boldsymbol{\delta} \succeq 0, \sum_{k=1}^{K} \delta_k \alpha_k \leq 1} g(\boldsymbol{\mu}, \boldsymbol{\delta}). \tag{42}$$

Similar algorithm like Algorithm 1 for Problem 1 can be readily developed for solving this problem. It can be shown that for the problem at hand, $g(\boldsymbol{\mu}, \boldsymbol{\delta})$ has sub-gradients, $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ for $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$, respectively, which are defined as $\theta_k = R_k - r'_k$ and $\zeta_k = \mathtt{Tr}(\boldsymbol{S}'_k)$, $k = 1, \ldots, K$, where $\{\boldsymbol{S}'_k\}$ and $\{r'_k\}$ are the Lagrangian minimizers that satisfy $\mathcal{L}(\{\boldsymbol{S}'_k\}, \{r'_k\}, \boldsymbol{\mu}, \boldsymbol{\delta}) = g(\boldsymbol{\mu}, \boldsymbol{\delta})$.

The convergence of the algorithm for this problem and the uniqueness of the solutions are similar as Algorithm 1. Let $\{\boldsymbol{S}^*_k\}$ and $\boldsymbol{\delta}^*$ denote the corresponding primal and dual optimal solutions. It is worth mentioning here that the obtained primal solution for this problem $P^*$ might not be equal to the user sum-power $\sum_{k=1}^{K} \mathtt{Tr}(\boldsymbol{S}^*_k)$ since some of the power constraints in (21), $\mathtt{Tr}(\boldsymbol{S}_k) \leq \alpha_k P$, may not be active in general.[4] Actually, $P^* = \sum_{k=1}^{K} \delta^*_k \mathtt{Tr}(\boldsymbol{S}^*_k)$ shown as follows. Since $P^* > 0$, it can be verified from (41) that $\sum_{k=1}^{K} \delta^*_k \alpha_k = 1$. Moreover, by the KKT conditions for the power constraints, $\sum_{k=1}^{K} \delta^*_k (\mathtt{Tr}(\boldsymbol{S}^*_k) - \alpha_k P^*) = 0$. Hence, $\sum_{k=1}^{K} \delta^*_k \mathtt{Tr}(\boldsymbol{S}^*_k) = \sum_{k=1}^{K} \delta^*_k \alpha_k P^* = P^*$.

The primal-dual approach used for solving the SPmin-PPC problem is also based on the connection between the power region and the corresponding capacity region, similar as that for the W-SPmin problem. However, their difference lies in that for the SPmin-PPC problem, with reference to Fig. 3, the weight vector for characterizing the solution on the boundary of the power region, $\boldsymbol{\lambda}$, and the weight vector for the corresponding capacity region, $\boldsymbol{\rho}$, are both unknown and, hence, they need to be found under the given power-profile vector $\boldsymbol{\alpha}$ as the corresponding optimal dual solution $\boldsymbol{\delta}^*$ and $\boldsymbol{\mu}^*$, respectively. In contrast, for the W-SPmin problem, only the unknown $\boldsymbol{\rho}$ needs to be found as the

[4]For example, in the case of two-user deterministic SISO-MAC as shown in Fig. 4 (a), if the given power-profile vector $\boldsymbol{\alpha}$ for Problem 3 is such that the intersected boundary power-tuple $(\alpha_1 P^*, \alpha_2 P^*)$ of the power region is located on the vertical (or horizontal) boundary segment of the power region, the solutions to Problem 3 will converge to the upper (or lower) vertex $(p^*_1, p^*_2)$ of the power region for which, clearly, $p^*_1 = \alpha_1 P^*$, but $p^*_2 < \alpha_2 P^*$. Thus, $P^* \neq p^*_1 + p^*_2$.

optimal dual solution $\boldsymbol{\mu}^*$ because $\boldsymbol{\lambda}$ is already given.

## V. Characterization of Power Region for TDMA

This section considers the characterization of the power region defined in (6) for the fading MIMO-MAC under TDMA, and for brevity only the W-SPmin problem (Problem 2) is investigated. The alternative means for characterizing the power region based on the power-profile vector, i.e., the SPmin-PPC problem under TDMA, is omitted since it can be readily obtained given the techniques developed in Section IV-B for the case of SDMA. The proposed algorithm for Problem 2 is also based on the Lagrange primal-dual approach.

First, it is noted that the constraints in (12) and (13) in Problem 2 can be combined and thus simplified, given the fact that in (4), the inequality constraints are always satisfied with equalities for power minimization. Hence, the Lagrangian of the primal problem can be written as

$$\mathcal{L}(\{\tau_k\},\{\boldsymbol{S}_k\},\boldsymbol{\mu}) = \sum_{k=1}^{K}\lambda_k \mathtt{Tr}(\boldsymbol{S}_k) - \sum_{k=1}^{K}\mu_k\left(\tau_k\mathbb{E}_\nu\left[\frac{1}{2}\log\left|\boldsymbol{H}_k(\nu)\frac{\boldsymbol{S}_k}{\tau_k}\boldsymbol{H}_k^\dagger(\nu)+\boldsymbol{I}\right|\right] - R_k\right), \tag{43}$$

where $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_K) \in \mathbb{R}_+^K$ denotes the dual variables associated with the constraints in (12). The variables, $\{\boldsymbol{S}_k\}$ and $\{\tau_k\}$, belong to the set denoted by $\mathcal{G}$, which is specified by the remaining constrains in (14), (15) and (16). Also note that the Lagrangian is a convex function of both $\{\boldsymbol{S}_k\}$ and $\{\tau_k\}$. By changing the variables as $\frac{\boldsymbol{S}_k}{\tau_k} \longmapsto \boldsymbol{W}_k, k=1,\ldots,K$, the Lagrangian can be rewritten as

$$\mathcal{L}(\{\tau_k\},\{\boldsymbol{W}_k\},\boldsymbol{\mu}) = \sum_{k=1}^{K}\lambda_k\tau_k \mathtt{Tr}(\boldsymbol{W}_k) - \sum_{k=1}^{K}\mu_k\left(\tau_k\mathbb{E}_\nu\left[\frac{1}{2}\log\left|\boldsymbol{H}_k(\nu)\boldsymbol{W}_k\boldsymbol{H}_k^\dagger(\nu)+\boldsymbol{I}\right|\right] - R_k\right). \tag{44}$$

For $k=1,\ldots,K$, define

$$z_k(\boldsymbol{W}_k,\mu_k) \triangleq \lambda_k\mathtt{Tr}(\boldsymbol{W}_k) - \mu_k\mathbb{E}_\nu\left[\frac{1}{2}\log\left|\boldsymbol{H}_k(\nu)\boldsymbol{W}_k\boldsymbol{H}_k^\dagger(\nu)+\boldsymbol{I}\right|\right], \tag{45}$$

and

$$\hat{z}_k(\mu_k) = \min_{\boldsymbol{W}_k\succeq 0} z_k(\boldsymbol{W}_k,\mu_k). \tag{46}$$

From (44), (45), and (46), the Lagrange dual function can be expressed as

$$g(\boldsymbol{\mu}) = \min_{\{\boldsymbol{W}_k,\tau_k\}\in\mathcal{G}} \mathcal{L}(\{\tau_k\},\{\boldsymbol{W}_k\},\boldsymbol{\mu}) \tag{47}$$

$$= \min_{\{\tau_k\}:\tau_k\geq 0\ \forall k,\sum_{k=1}^K\tau_k=1} \sum_{k=1}^{K}\tau_k\hat{z}_k(\mu_k) + \sum_{k=1}^{K}\mu_k R_k. \tag{48}$$

The optimal value of the primal problem, denoted by $q^*$, can be then obtained as

$$q^* = \max_{\boldsymbol{\mu}\succeq 0} g(\boldsymbol{\mu}) \tag{49}$$

$$\triangleq \sum_{k=1}^{K} \tau_k^* \hat{z}_k(\mu_k^*) + \sum_{k=1}^{K} \mu_k^* R_k. \tag{50}$$

*Theorem 4:* If $\{\boldsymbol{S}_k^*\}, \{\tau_k^*\}, \{r_k^*\}$ are the optimal primal solutions and $\{\mu_k^*\}$ are the optimal dual solutions for Problem 2 under the strictly positive weight vector $\boldsymbol{\lambda}$ and rate target $\boldsymbol{R}$, they must satisfy

$$\lambda_k p_k^* - \mu_k^* r_k^* = c^* \tau_k^*, \qquad k = 1, \ldots, K, \tag{51}$$

where $p_k^* = \mathtt{Tr}(\boldsymbol{S}_k^*)$, $r_k^* = \tau_k^* \mathbb{E}_\nu \left[\frac{1}{2}\log\left|\boldsymbol{H}_k(\nu)\frac{\boldsymbol{S}_k^*}{\tau_k^*}\boldsymbol{H}_k^\dagger(\nu) + \boldsymbol{I}\right|\right] = R_k$, and $c^*$ is a constant.

*Proof:* Since $\boldsymbol{R}$ and $\boldsymbol{\lambda}$ are both strictly positive, then so are the obtained solutions $\{\tau_k^*\}$. As a result, it is necessary to have $\hat{z}_1(\mu_1^*) = \cdots = \hat{z}_K(\mu_K^*)$, otherwise the minimization in (48) must lead to only one user assigned with the total time slot, i.e., $\tau_{k'}^* = 1$ and $\tau_k^* = 0, k \neq k'$ where $k' = \arg\min_k \hat{z}_k(\mu_k^*)$. Using this equality and also (45), (46), the proof is completed. ■

Using Theorem 4, the algorithm for Problem 2 can be obtained as follows: In each iteration, the algorithm updates the dual variables $\boldsymbol{\mu}$ such that $\hat{z}_1(\mu_1) = \cdots = \hat{z}_K(\mu_K) = c$. It then checks whether the obtained rates can support more than the target rates, and increases $c$ if they do or decreases it otherwise in the next iteration, until the rate targets are exactly met and $c$ converges to $c^*$. The details for the proposed algorithm are presented below.

*Algorithm 2:*

- **Given** $c_{\min} \leq c^* \leq c_{\max}$.
- **Repeat**
  1. $c \leftarrow \frac{1}{2}(c_{\min} + c_{\max})$.
  2. For each $k$, obtain the optimal solutions $\boldsymbol{W}'_k$ and $\mu'_k$ such that $\hat{z}_k(\mu'_k) = z_k(\boldsymbol{W}'_k, \mu'_k) = c$.[5] Do the above for $k = 1, \ldots, K$.
  3. Compute $\tau'_k$ such that $R_k = \tau'_k \mathbb{E}_\nu \left[\frac{1}{2}\log\left|\boldsymbol{H}_k(\nu)\boldsymbol{W}'_k\boldsymbol{H}_k^\dagger(\nu) + \boldsymbol{I}\right|\right]$ for $k = 1, \ldots, K$.
  4. If $\sum_{k=1}^{K} \tau'_k < 1$, $c_{\min} \leftarrow c$; otherwise $c_{\max} \leftarrow c$.

[5]For each given $\mu_k$, $\hat{z}_k(\mu_k)$ can be obtained by minimizing $z_k(\boldsymbol{W}_k, \mu_k)$ over $\boldsymbol{W}_k$ as in (46) by means of a convex optimization method, e.g., the interior-point method [19]. The quantity $\mu'_k$ for which $\hat{z}_k(\mu'_k) = c$ can then be obtained by a bisection search over $\mu_k$ using the fact that $\hat{z}_k(\mu_k)$ is a decreasing function of $\mu_k$.

- **Until** $c_{\max} - c_{\min} < \delta$ where $\delta$ is a small positive constant that controls the algorithm accuracy.

Since $\hat{z}_k(\mu_k^*) \leq 0, \forall k$, it follows that $c^* \leq 0$. Thus, we can take $c_{\max} = 0$. Similar as Appendix I, we can obtain the upper bounds $\mu_k^{(0)}$'s on the optimal dual solutions $\mu_k^*$'s. From (43) and using the fact that $g(\boldsymbol{\mu}^*) \geq 0$, it is easy to show that $c^* \geq -\sum_{k=1}^K \mu_k^* R_k$. Thus, we can take $c_{\min} = -\sum_{k=1}^K \mu_k^{(0)} R_k$. At last, the convergence of the above algorithm as well as the uniqueness of the obtained solutions are ensured by the uniqueness of $c^*$ in Theorem 4.

## VI. Numerical Results

This section presents the power region for a fading MIMO-MAC with $r = 2$ receive antennas and $K = 2$ MTs each equipped with $t_1 = t_2 = 2$ transmit antennas. It is assumed that the receive antennas at the BS are sufficiently separated that they experience independent fading, while the fading levels are correlated across the transmit antennas because of their realistic size limitations. Under this assumption, the employed channel model for MT $k$ is given by $\boldsymbol{H}_k(\nu) = \boldsymbol{H}_w(\nu)\boldsymbol{Q}_k^{1/2}$ for $k = 1, 2$, where $\boldsymbol{Q}_k \in \mathbb{C}^{t_k \times t_k}$ denotes the transmit antenna correlation matrix for MT $k$ and is assumed to be constant over all fading states of $\nu$. $\boldsymbol{H}_w(\nu) \in \mathbb{C}^{r \times t_k}$ denotes the Rayleigh-fading channel matrix that is independent across two MTs and across all fading states, and has independent entries distributed as $\mathcal{CN}(0, 1)$. Similar as the proof given in [11] and [12], it can be shown that the expressions in (38) and (45), for SDMA and TDMA, respectively, are both minimized when the transmit signal covariance matrix, $\boldsymbol{S}_k$, has the same set of eigenvectors as $\boldsymbol{Q}_k$, $k = 1, 2$, i.e., if $\boldsymbol{A}_k \boldsymbol{\Lambda}_k \boldsymbol{A}_k^\dagger$ is the eigenvalue decomposition of $\boldsymbol{Q}_k$, the optimal $\boldsymbol{S}_k$ then takes the form of $\boldsymbol{A}_k \boldsymbol{\Sigma}_k \boldsymbol{A}_k^\dagger$ for some diagonal matrix $\boldsymbol{\Sigma}_k$. This observation reduces the number of (real) variables from $t_k^2$ in $\boldsymbol{S}_k$ to $t_k$ in $\boldsymbol{\Sigma}_k$ for MT $k$ and in turn reduces the total algorithm complexity.[6] Monte-Carlo simulation with 5000 independent realizations of the random channels is used to approximate the actual expectation over the fading states. The simulation assumes that the target rate is $\boldsymbol{R} = [2\ 1]^T$ nats/sec/Hz for two MTs, and

$$\boldsymbol{Q}_1 = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \end{bmatrix}, \qquad \boldsymbol{Q}_2 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

[6]The proposed algorithms work for all kinds of CDI, e.g., with constant channel mean matrix, constant channel covariance matrix, or combinations of them in all general forms. However, for most of these cases, a similar variable-reduction like in this numerical example is not possible.

Fig. 6 and Fig. 7 show the power region for this fading MAC under SDMA and TDMA obtained by solving Problem 1 and Problem 2, respectively.

For the SDMA case as is shown in Fig. 6, there are two corner points on the boundary of the power region, denoted by point A and B, which can be obtained by the greedy algorithm described in Section IV-A. Recall that in this greedy algorithm, the BS first picks one possible decoding order for the MTs, and then starting from the last decoded MT, it minimizes the power required to maintain the target rate for each MT while considering the MTs that have yet been decoded as interference. In this figure, the power-pair A corresponds to the decoding order $2 \rightarrow 1$, while the power-pair B does for the reversed decoding order. The greedy algorithm achieves the optimal power-pairs for the W-SPmin problem under some weight vectors, e.g., point A for $\lambda_1 \gg \lambda_2$ and point B for $\lambda_2 \gg \lambda_1$, but might be suboptimal for weight vectors other than these extreme choices. For example, the minimum value of $0.4p_1 + 0.6p_2$ to support the target rate $[2 \ 1]^T$ is 11.5 while the greedy algorithm leads to 12.8 and 13.3 units of power for power-pair A and B, respectively. Moreover, the boundary curve of the power region is not attainable by simply time-sharing these two corner points, as shown by the dashed line in Fig.6.

The power region under TDMA for this MAC is shown in Fig. 7. For comparison, the power region under SDMA is also included in this figure. This figure uses the log scale for the powers and, hence, the power region under SDMA looks as a non-convex set. The power savings achieved by SDMA are observed to be substantial compared to TDMA, even in this case where the number of transmit antennas at each MT is equal to that of the receive antennas at the BS, i.e., both SDMA and TDMA have the same number of degrees of transmission freedom in the spatial domain, which is two in this case. The power-pairs on the boundary of the power region under TDMA correspond to different time-slot durations, $\tau_k$, $k = 1, 2$, assigned to each MT. The power-pair A shown in Fig. 7 is achieved by assigning equal duration of time slot for both MTs, i.e., $\tau_1 = \tau_2 = 0.5$ as in the conventional TDMA. Clearly, this power-pair is optimal for the W-SPmin problem under a unique weight vector $\boldsymbol{\lambda}$, and for the SPmin-PPC problem under a unique power-profile vector $\boldsymbol{\alpha}$, but it is suboptimal in all the other cases. For example, the minimum sum-power $p_1 + p_2$ to achieve the target rate $[2 \ 1]^T$ is 42 units of power for $\tau_1 = 0.66$ and $\tau_2 = 0.34$, as compared to 69 units of power for $\tau_1 = \tau_2 = 0.5$.

## VII. Conclusions

This paper characterizes the power region for the fading MIMO-MAC. Motivated by a general relationship between the power region and the corresponding capacity region, the Lagrange primal-dual approach is employed to characterize all pareto optimal power-tuples on the boundary of the power region. These optimal power-tuples provide different power tradeoff among the MTs and also ensure the fairness of power consumption among them. The algorithms developed in this paper can be used in the wireless cellular network for the BS to control the transmit powers from the MTs in the uplink transmission. Two multiple access techniques, namely, SDMA and TDMA, are considered in this paper. It is observed that substantial power savings can be obtained by using SDMA compared to TDMA. This observation provides an important information-theoretic guidance for practical system designs, i.e., if the complexity for implementing the optimal SDMA can be tailed for, an enormous capacity gain is still possible over the conventional TDMA-based network. The multiuser transmit-covariance feedback scheme studied in this paper optimizes the transmit covariance matrices of all the MTs based on their long-term CDI. Hence, this scheme reduces significantly the feedback complexity compared to other feedback schemes based on the instantaneous channel realizations. As a result, this scheme is practically suitable for wireless channels that exhibit some consistent long-term channel statistics. The results obtained in this paper can provide insightful guidelines to many applications in wireless networks including resource allocation, partial channel feedback, and multiuser space-time code design.

## Appendix I
## Initial Ellipsoid for Algorithm 1

In the appendix, one possible method to obtain the initial ellipsoid $\mathcal{E}[0]$ for Algorithm 1 is presented. First, we obtain an upper bound $\mu_j^{(0)}$ on $\mu_j^*$ for any given $j \in \{1, \ldots, K\}$. Let $\{\boldsymbol{S}_k^{(j)}\} \in \mathcal{D}$ be any set of transmit covariance matricies that achieve $\{r_k^{(j)}\}$ given by

$$r_k^{(j)} = \begin{cases} R_k^* & k \neq j \\ R_k^* + 1 & k = j. \end{cases} \tag{52}$$

From the definition of the dual function given by (29), we have

$$g(\boldsymbol{\mu}^*) \leq \mathcal{L}(\{\boldsymbol{S}_k^{(j)}\}, \{r_k^{(j)}\}, \boldsymbol{\mu}^*) = \sum_{k=1}^{K} \lambda_k \mathtt{Tr}\left(\boldsymbol{S}_k^{(j)}\right) - \mu_j^*.$$

Since $g(\boldsymbol{\mu}^*) \geq 0$, it follows that

$$\mu_j^* \leq \sum_{k=1}^{K} \lambda_k \texttt{Tr}\left(\boldsymbol{S}_k^{(j)}\right). \tag{53}$$

Thus, $\mu_j^{(0)} = \sum_{k=1}^{K} \lambda_k \texttt{Tr}\left(\boldsymbol{S}_k^{(j)}\right)$. Similar upper bounds can be found for all other $j$. Next, $\mathcal{E}[0]$ can be chosen to cover the hyper-cube in $\mathbb{R}^K$ specified by $\mu_j^{(0)}$'s.

## APPENDIX II
## PROOF OF THEOREM 3

This Appendix proves the uniqueness of the solutions for the optimal transmit covariance matrices $\{\boldsymbol{S}_k^*\}$ in Problem 1. Without loss of generality, it is assumed that $\mu_1^* \geq \cdots \geq \mu_K^* > \mu_{K+1}^* = 0$. If $\{\boldsymbol{S}_k^{(1)}\}$ and $\{\boldsymbol{S}_k^{(2)}\}$ are two sets of optimal solutions for Problem 1, from (28) and by using Theorem 2, it follows that

$$q^* = \sum_{k=1}^{K} \lambda_k \texttt{Tr}(\boldsymbol{S}_k^{(j)}) + \sum_{k=1}^{K} \mu_k^* R_k - \sum_{k=1}^{K} \left(\mu_k^* - \mu_{k+1}^*\right) \mathbb{E}_\nu \left[\frac{1}{2}\log\left|\sum_{i=1}^{k} \boldsymbol{H}_i(\nu)\boldsymbol{S}_i^{(j)}\boldsymbol{H}_i(\nu)^\dagger + \boldsymbol{I}\right|\right], \tag{54}$$

for $j = 1, 2$. Since the problem at hand is convex, for any $\beta \in [0, 1]$, $\boldsymbol{S}_k = \beta\boldsymbol{S}_k^{(1)} + \bar{\beta}\boldsymbol{S}_k^{(2)}$ is also an optimal solution to satisfy (54), where $\bar{\beta} = 1 - \beta$. This fact together with the concavity of the $\log|\cdot|$ function implies that

$$\mathbb{E}_\nu\left[\log\left|\beta\boldsymbol{A}^{(1)}(\nu) + \bar{\beta}\boldsymbol{A}^{(2)}(\nu)\right| - \beta\log\left|\boldsymbol{A}^{(1)}(\nu)\right| - \bar{\beta}\log\left|\boldsymbol{A}^{(2)}(\nu)\right|\right] = 0, \tag{55}$$

where $\boldsymbol{A}^{(j)}(\nu) \triangleq \sum_{k=1}^{K} \boldsymbol{H}_k(\nu)\boldsymbol{S}_k^{(j)}\boldsymbol{H}_k(\nu)^\dagger + \boldsymbol{I}$. Let $f(\beta)$ denote the function on the LHS of the above equation, then $f(\beta) = 0$, for all $0 \leq \beta \leq 1$. Because $f(\beta)$ is twice continuously differentiable, both of its first and second derivatives must vanish, i.e.,

$$\frac{d^2 f(\beta)}{d\beta^2} = -\mathbb{E}_\nu\left[\texttt{Tr}\left(\left(\left(\boldsymbol{A}^{(1)}(\nu) - \boldsymbol{A}^{(2)}(\nu)\right)\left(\beta\boldsymbol{A}^{(1)}(\nu) + \bar{\beta}\boldsymbol{A}^{(2)}(\nu)\right)^{-1}\right)^2\right)\right] = 0, \quad \forall\beta. \tag{56}$$

For every $\nu$, the matrix in $\texttt{Tr}(\cdot)$ of the above equation is a positive semi-definite matrix and, hence, it has a nonnegative trace. Since the expectation of a nonnegative random variable is zero, it must be zero *a.s.*, or $\boldsymbol{A}^{(1)}(\nu) = \boldsymbol{A}^{(2)}(\nu)$ *a.s.*, which implies that $\boldsymbol{S}_k^{(1)} = \boldsymbol{S}_k^{(2)}$.

# REFERENCES


[1] I. E. Telatar, “Capacity of multi-antenna Gaussian channels,” *Eur. Trans. Telecommun.*, vol. 10, no. 6, pp. 585-595, Nov. 1999.

[2] G. J. Foschini and M. J. Gans, “On limits of wireless communications in a fading environment when using multiple antennas,” *Wireless Personal Commun.*, vol. 6, no. 3, pp. 311-335, Mar. 1998.

[3] G. G. Raleigh and J. M. Cioffi, “Spatial-temporal coding for wireless communications,” *IEEE Trans. Commun.*, vol. 46, no. 3, pp. 357-366, Mar. 1998.

[4] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, “Capacity limits of MIMO channels,” *IEEE J. Sel. Areas Commun.*, vol. 21, no. 5, pp. 684-702, Jun. 2003.

[5] E. Visotsky and U. Madhow, “Space-time transmit precoding with imperfect feedback,” *IEEE Trans. Inf. Theory,* vol. 47, no. 6, pp. 2632-2639, Sep. 2001.

[6] S. A. Jafar and A. Goldsmith, “Transmitter optimization and optimality of beamforming for multiple antenna systems with imperfect feedback,” *IEEE Trans. Wireless Commun.*, vol. 3, no.4, pp. 1165-1175, Jul. 2004.

[7] S. Simon and A. Moustakas, “Optimizing MIMO antenna systems with channel covariance feedback,” *IEEE J. Select. Areas Commun.*, vol. 21, pp. 406-417, Apr. 2003.

[8] E. Jorswieck and H. Boche, “Channel capacity and capacity-range of beamforming in MIMO wireless systems under correlated fading with covariance feedback,” *IEEE J. Select. Areas Commun.*, vol. 3, no. 5, pp. 1543-1553, Sep. 2004.

[9] D. J. Love, R. W. Heath Jr, W. Santipachz, and M. L. Honigz, “What is the value of limited feedback for MIMO channels?” *IEEE Commun. Magazine*, pp. 54-59, Oct. 2004.

[10] R. Zhang, Y. C. Liang, R. Narasimhan, and J. M. Cioffi, “Approaching MIMO-OFDM capacity with per-antenna power and rate feedback,” *IEEE J. Sel. Areas Commun.*, vol. 25, no. 7, pp. 1284-1297, Sep. 2007.

[11] S. A. Jafar, S. Vishwanath, and A. J. Goldsmith, “Vector MAC capacity region with covariance feedback,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, p. 321, Jun. 2001.

[12] A. Soysal and S. Ulukus, “Transmit directions and optimality of beamforming in MIMO-MAC with partial CSI at the transmitters,” in *Proc. Conference on Inform., Science, and Systems (CISS)*, Mar. 2005.

[13] D. Tse and S. Hanly,“Multi-access fading channels-Part I: polymatroid structure, optimal resource allocation and throughput capacities,” *IEEE Trans. Inf. Theory*, vol. 44, no. 7, pp. 2796-2815, Nov. 1998.

[14] S. Hanly and D. Tse,“Multi-access fading channels-Part II: Delay-limited capacities,” *IEEE Trans. Inf. Theory*, vol. 44, no. 7, pp. 2816-2831, Nov. 1998.

[15] M. Mecking, “Minimizing transmit power for fading multiple-access channels,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, p. 494, Jun. 2000.

[16] A. Lampe, R. Müller, and J. Huber, “Transmit power allocation for Gaussian multiple-access channels with diversity,” in *Proc. IEEE Inf. Theory Workshop (ITW)*, p. 101, Jun. 1999.

[17] R. D. Yates, “Iterative multiuser uplink and downlink beamforming under SINR constraints,” *IEEE J. Sel. Areas Commun.*, vol. 13, no. 7, pp. 1341-1347, Sep. 1995.

[18] F. Rashid-Farrokhi, L. Tassiulas, and K. Liu, “Joint optimal power control and beamforming in wireless networks using antenna arrays,” *IEEE Trans. Commun.*, vol. 46, pp. 1313-1323, Oct. 1998.

[19] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[20] T. Cover and J. Thomas, *Elements of information theory,* New York: Wiley, 1991.

[21] E. Jorswieck and H. Boche, ”Rate balancing for the multi-antenna Gaussian broadcast channel,” in *Proc. IEEE Int. Symp. Spread-Spectrum Techniques and Applic.*, pp. 545-549, 2002.

[22] H. Viswanathan, S. Venkatesan and H. C. Huang, “Downlink capacity evaluation of cellular networks with known interference cancellation,” *IEEE J. Sel. Areas Commun.*, vol. 21, pp. 802-811, Jun. 2003.

[23] J. Oh, S. J. Kim, R. Narasimhan and J. M. Cioffi, ”Transmit power optimization for Gaussian vector broadcast channels,” in *Proc. IEEE Conf. Comm. (ICC)*, Seoul, May 2005.

[24] R. G. Bland, D. Goldfarb, and M. J. Todd, “The ellipsoid method: A survey,” *Operations Research,* vol. 29, no. 6, pp. 1039-1091, 1981.

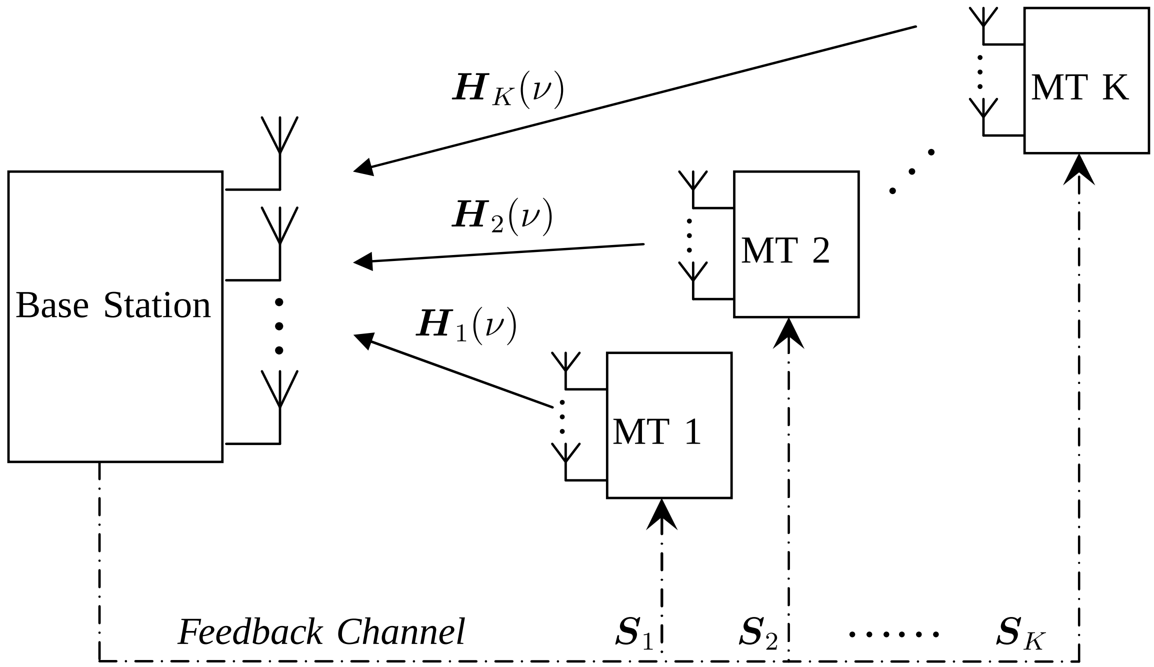


Fig. 1. The fading MIMO-MAC with the multiuser transmit-covariance feedback.

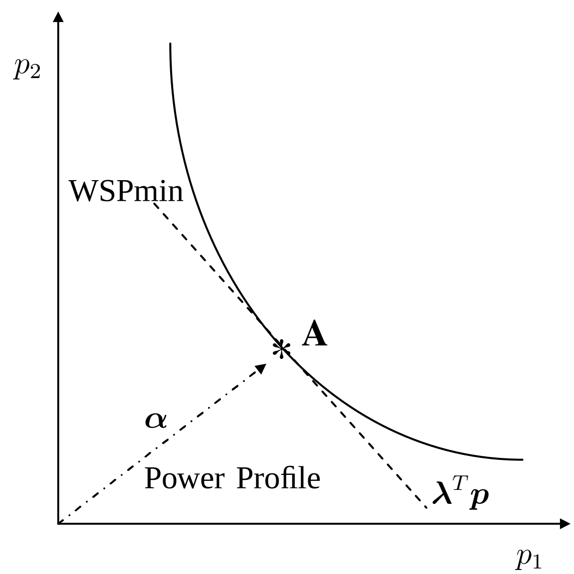


Fig. 2. Characterization of power region via WSPmin or power-profile vector.

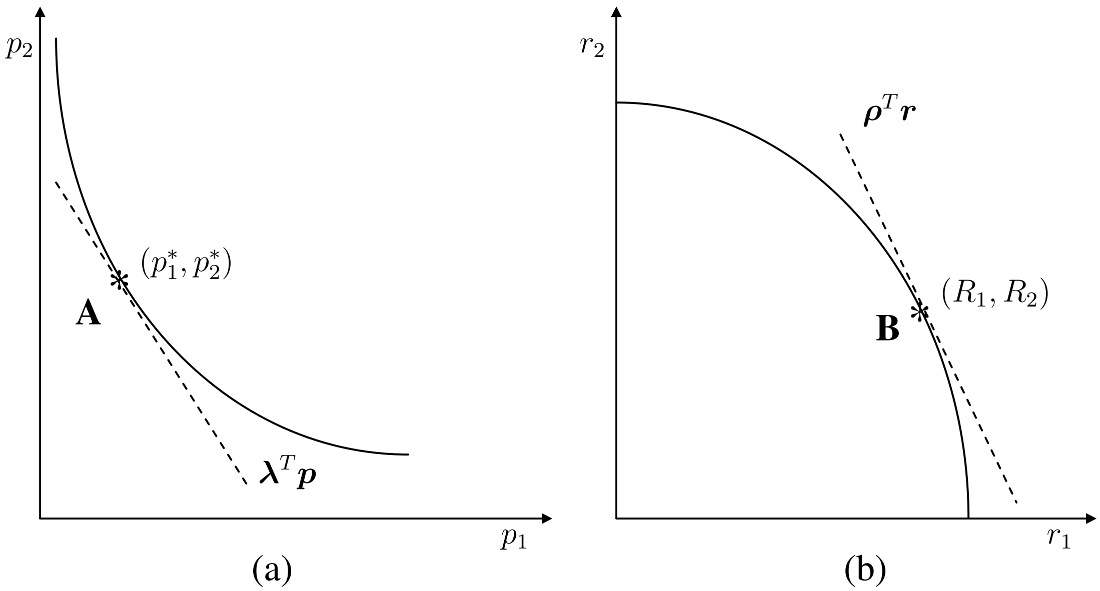


Fig. 3. The relationship between each boundary point of the power region and its corresponding capacity region for a two-user fading MAC. (a) Power region under the rate constraint $(R_1, R_2)$, where $(p_1^*, p_2^*)$, represented by point A, is on its boundary and achieves the minimum weighted sum-power under the weights $\lambda_1$ and $\lambda_2$, i.e., $\lambda_1 p_1^* + \lambda_2 p_2^* = p^*$; (b) Corresponding capacity region of $(p_1^*, p_2^*)$ under the weighted sum-power constraint $\lambda_1 p_1 + \lambda_2 p_2 \leq p^*$, where $(R_1, R_2)$, represented by point B, is on its boundary. Moreover, $(R_1, R_2)$ maximizes the weighted sum-rate $\rho_1 r_1 + \rho_2 r_2$ for some nonnegative weight vector $\boldsymbol{\rho}$.

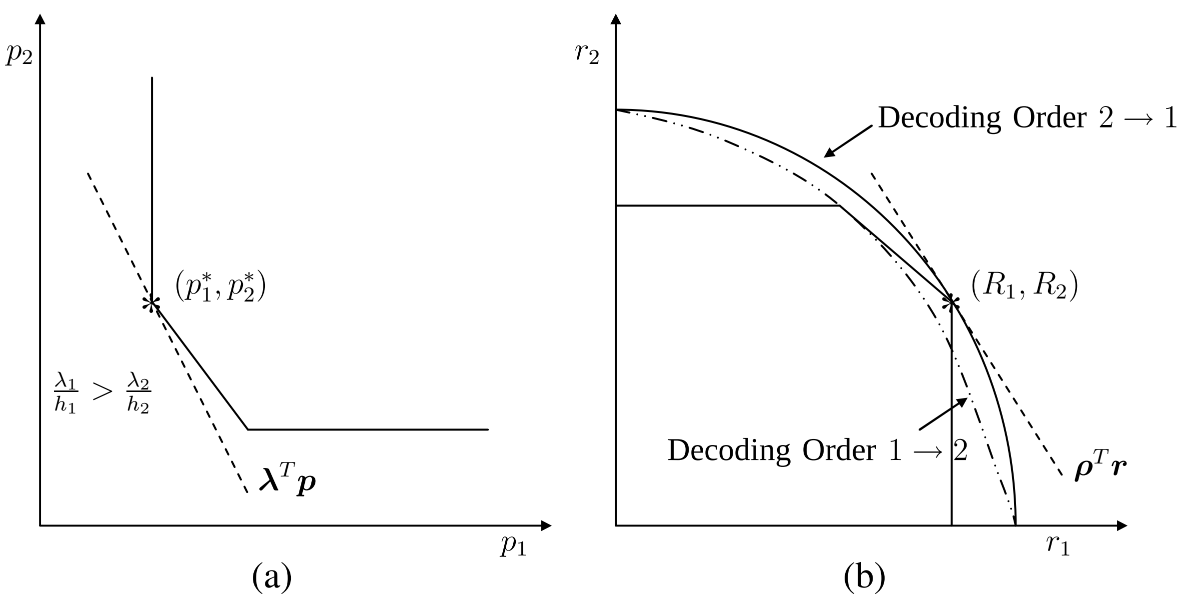


Fig. 4. The relationship between each boundary point of the power region and its corresponding capacity region for a two-user deterministic SISO-MAC under SDMA. (a) Power region under the rate constraint $(R_1, R_2)$, where $(p_1^*, p_2^*)$ is on its boundary and achieves the minimum weighted sum-power for the weights $\lambda_1$ and $\lambda_2$, and the associated decoding order is $2 \rightarrow 1$ since $\frac{\lambda_1}{h_1} > \frac{\lambda_2}{h_2}$; (b) Corresponding capacity region of $(p_1^*, p_2^*)$ under the weighted sum-power constraint $\lambda_1 p_1 + \lambda_2 p_2 \leq p^*$, where $p^* = \lambda_1 p_1^* + \lambda_2 p_2^*$.

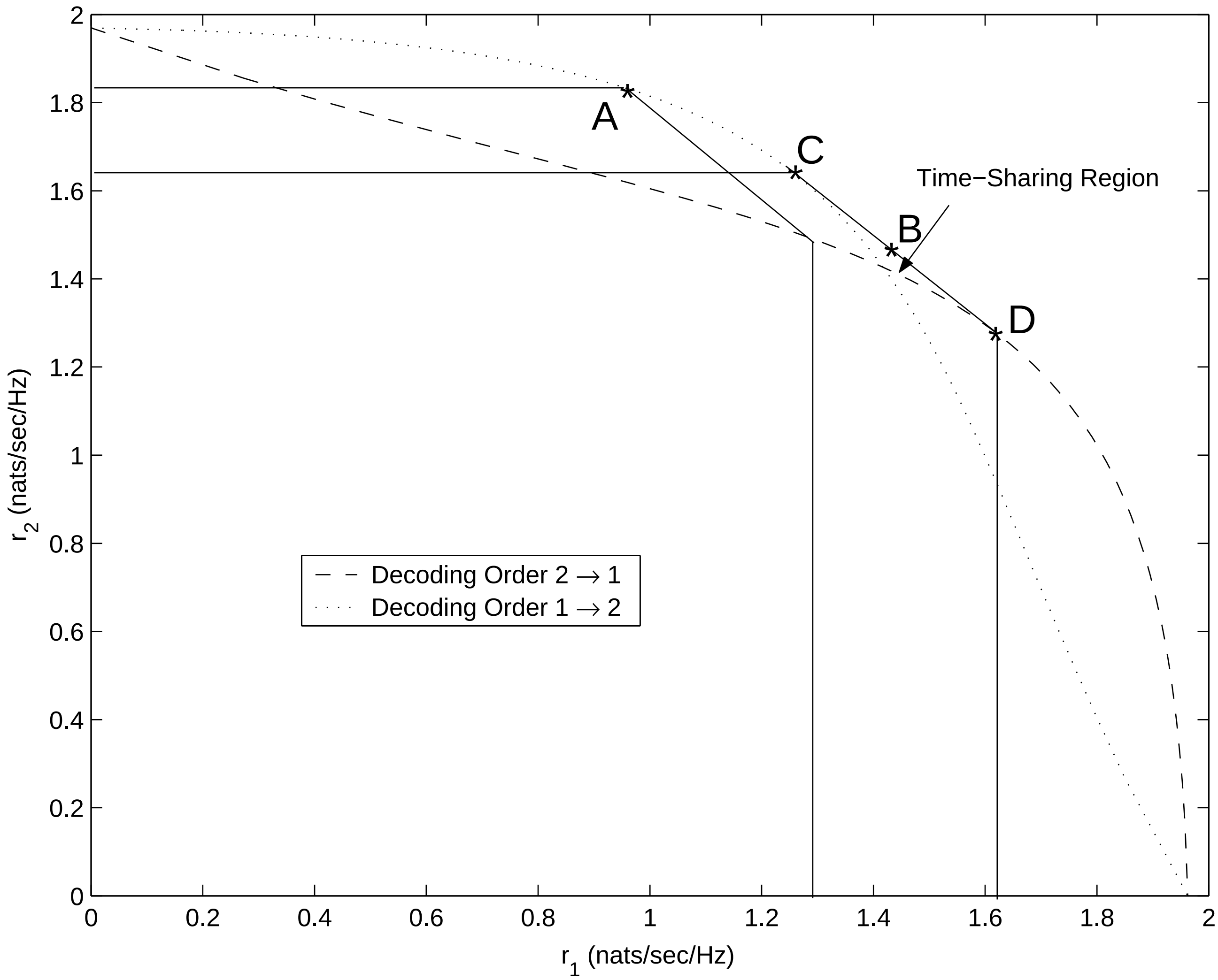


Fig. 5. Capacity region for a two-user symmetric-fading SIMO-MAC under a sum-power constraint: $p_1 + p_2 \leq 10$. The multiple access technique is SDMA, and $t_1 = t_2 = 1$, $r = 2$.

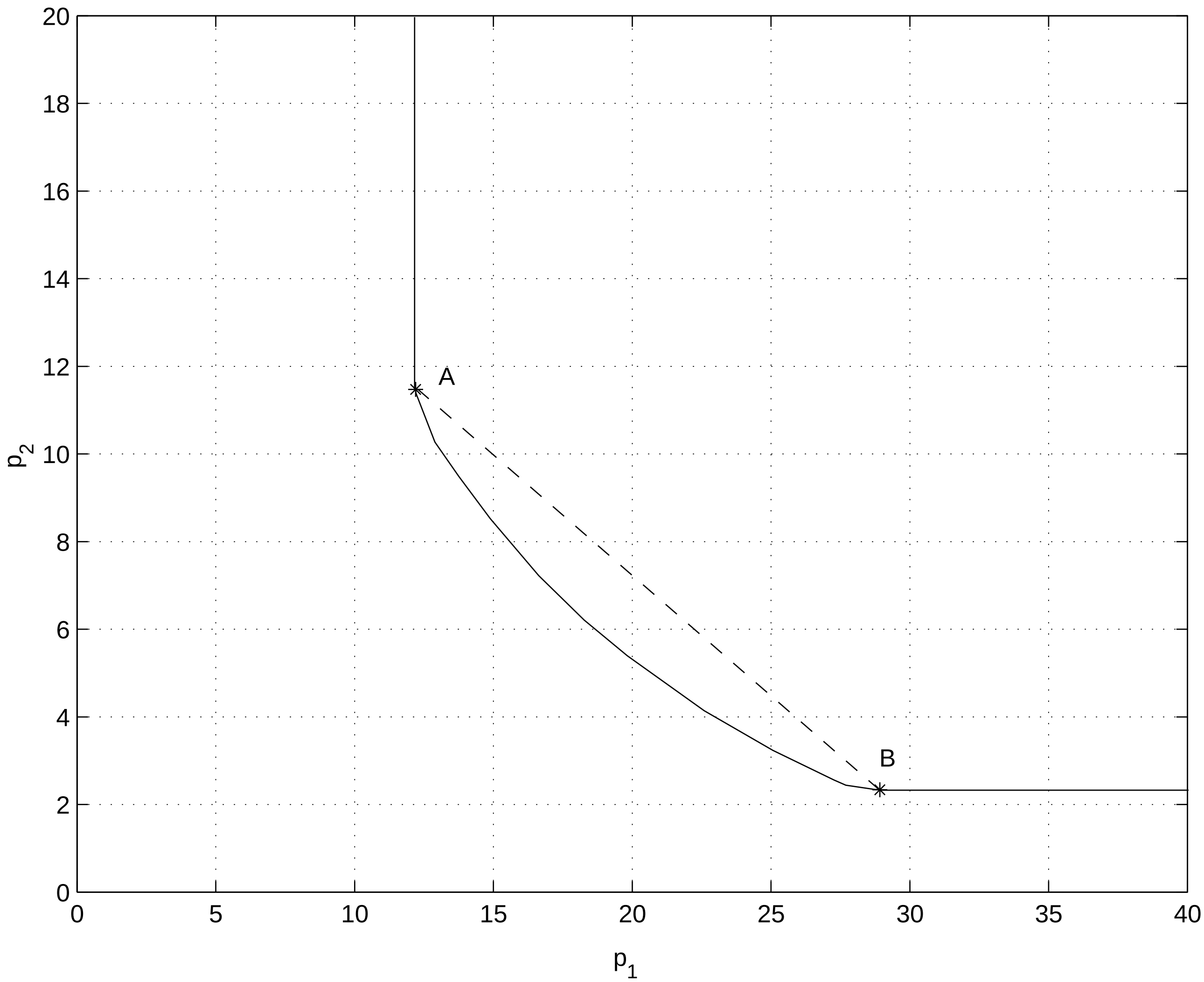


Fig. 6. Power region for a two-user transmit-correlated fading MIMO-MAC under SDMA with $t_1 = t_2 = 2$, $r = 2$, and the target rate, $\boldsymbol{R} = [2\ 1]^T$ nats/sec/Hz.

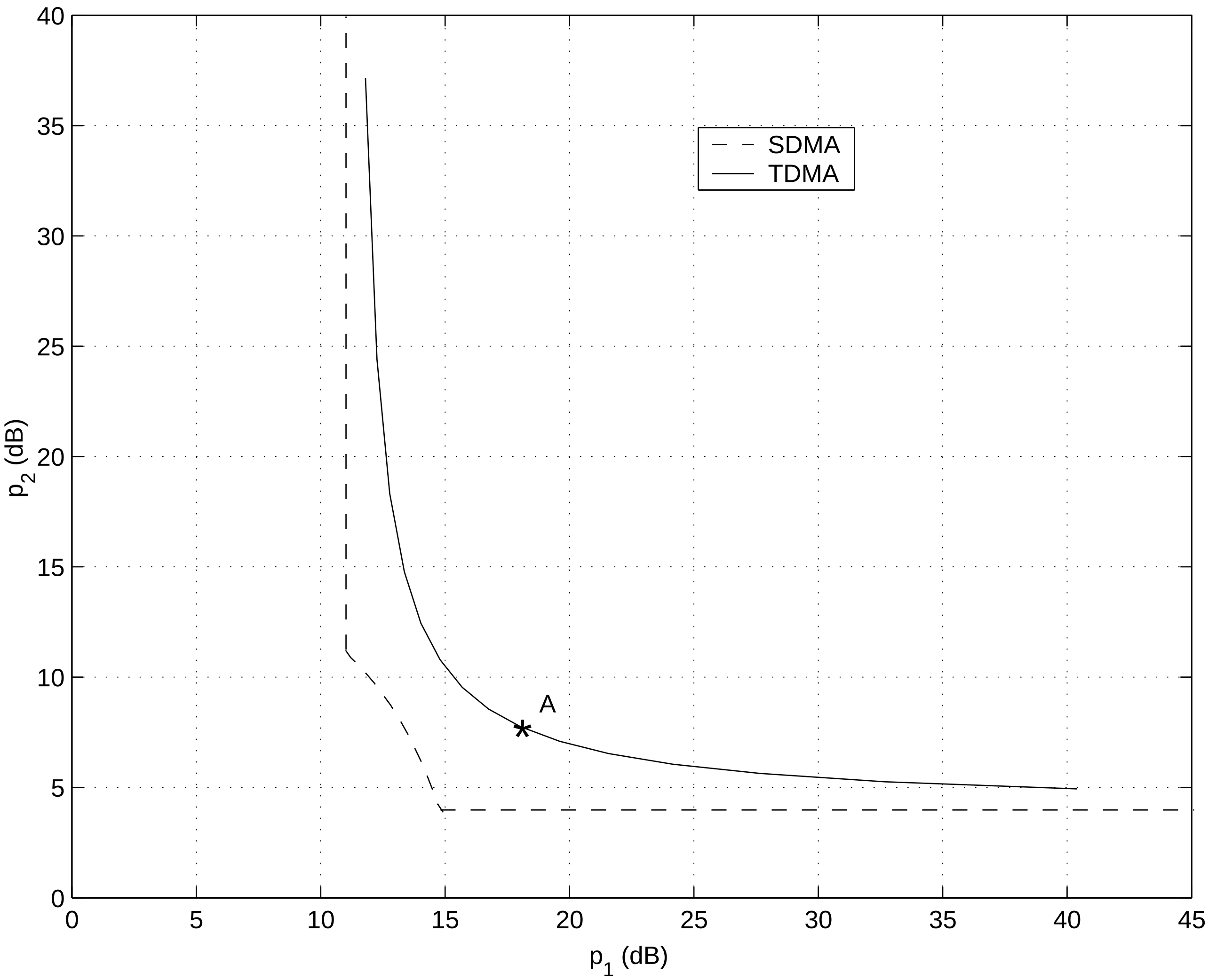


Fig. 7. Power region for a two-user transmit-correlated fading MIMO-MAC under SDMA and TDMA with $t_1 = t_2 = 2$, $r = 2$, and the target rate, $\boldsymbol{R} = [2\ 1]^T$ nats/sec/Hz.